S.-T. Yau College Student Mathematics Contests 2018

Analysis and Differential Equations

Individual

Please solve the following problems.

1. A map $f : A \to \mathbb{R}^n$, $A \subset \mathbb{R}^m$ is a $(L-)$Lipschitz map if there is a constant $L < \infty$ such that

$$|f(x) - f(y)| \leq L|x - y| \text{ for all } x, y \in A.$$

The smallest such constant L is called the Lipschitz constant of f and denoted by $\mathrm{Lip}(f)$.

1) Let $f : A \to \mathbb{R}, A \subset \mathbb{R}^m$. Let $G = \{(x, f(x)) : x \in A\} \subset \mathbb{R}^m \times \mathbb{R}$ denote its graph. Suppose that there exists $\alpha > 0$, such that for any $z \in G$, the cone

$$C_{z,\alpha} := z + \{(y_1, y_2) \in \mathbb{R}^m \times \mathbb{R} : |y_2| > \alpha |y_1|\} \subset \mathbb{R}^m \times \mathbb{R}$$

does not meet G. Prove that f is Lipschitz, with $\mathrm{Lip}(f) \leq \alpha$.

2) Suppose that $f : A \to \mathbb{R}$, $A \subset \mathbb{R}^m$ is an L-Lipschitz map. Prove that f can be extended to a L-Lipschitz map on $\mathbb{R}^m$, that is, there exists an L-Lipschitz map $g : \mathbb{R}^m \to \mathbb{R}$, such that $g\lfloor_A = f$.

2. Let D be a region in the complex plane, z_0 a point in D. Let U be the open unit disk. If D is simply connected and $D \neq C$, then there exists at least one univalent function from $f : D \to U$ such that $f^{'}(z_0) > 0$.

You can not use Riemann mapping theorem directly.

3. Let $P(x)$ be polynomial of degree n. Show that

$$|P(0)| \leq C \int_{-1}^{1} |P(x)| dx$$

4. Prove uniqueness of solutions to the following problem

$$\Delta u + \sqrt{u} = 0 \text{ in } \Omega$$

$$u > 0 \text{ in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

5. Let ℓ_2 be the Hilbert space of all square summable complex sequences $x = (x_k)_{k\geq 1}$, equipped with the following inner product and norm

$$\langle x,\, y\rangle = \sum_{k\geq 1} x_k\, \overline{y_k} \quad \text{and} \quad \|x\| = \Big(\sum_{k\geq 1} |x_k|^2\Big)^{1/2}.$$

Let $u : \ell_2 \to \ell_2$ be the linear operator defined by

$$\forall\, x = (x_k)_{k\geq 1} \in \ell_2, \quad u(x) = \Big(\sum_{k=1}^{\infty} \frac{x_k}{j+k}\Big)_{j\geq 1}.$$

The aim of this exercise is to calculate the norm $\|u\|$.

a) Let φ be the 2π-periodic function defined by $\varphi(t) = \mathrm{i}(\pi - t)$ for $0 \leq t < 2\pi$, where $\mathrm{i} = \sqrt{-1}$. Show that

$$\sum_{j,k\geq 1} \frac{x_j y_k}{j+k} = \frac{1}{2\pi}\int_0^{2\pi} \Big(\sum_{j\geq 1} x_j e^{-\mathrm{i}jt}\Big)\Big(\sum_{k\geq 1} y_k e^{-\mathrm{i}kt}\Big)\varphi(t)dt\,, \quad \forall\, x, y \in \ell_2.$$

b) Deduce that u is bounded and $\|u\| \leq \pi$.

c) For any given $n \geq 1$ let $a_n = (1, \frac{1}{\sqrt{2}}, \cdots, \frac{1}{\sqrt{n}}, 0, 0, \cdots)$. Show that

$$\langle u(a_n),\, a_n\rangle \geq \pi \ln n + \mathrm{O}(1).$$

Deduce $\|u\| \geq \pi$.

d) For any given $n \geq 1$ let $a_n = (1, \frac{1}{\sqrt{2}}, \cdots, \frac{1}{\sqrt{n}}, 0, 0, \cdots)$. Show that

$$\langle u(a_n),\, a_n\rangle \geq \pi \ln n + \mathrm{O}(1).$$

Deduce $\|u\| \geq \pi$.